arXiv:math-ph/0309037v1 15 Sep 2003

The Moyal Bracket in the Coherent States framework

M. Daoud[1*] and E. H. El Kinani[2*]

The Abdus Salam International Centre for Theoretical Physics, ICTP-Strada costiera 11, 34100 Tieste Italy.

Abstract.

The star product and Moyal bracket are introduced using the coherent states corresponding to quantum systems with non-linear spectra. Two kinds of coherent state are considered. The first kind is the set of Gazeau-Klauder coherent states and the second kind are constructed following the Perelomov-Klauder approach. The particular case of the harmonic oscillator is also discussed.

[(2)]E-mail: hkinani@ictp.trieste.it or el-kinani@fste.ac.ma

[(*)]Permanent address:

[(1)] LPMC, Département de Physique, Faculté des Sciences B.P 28/S, Université Ibn Zohr Agadir, Morocco

[(2)] GMP, Département de Mathématiques, Faculté des Sciences et Technique, Boutalamine B.P 509, Errachidia, Morocco.

1 Introduction

In classical mechanics, observables are smooth functions on phase space, which constitute a Poisson algebra, while in quantum mechanics, the observables constitute a noncommutative associative algebra. Deformation quantization is the basis of one of the important attempts aiming to construct a quantum system starting from a classical mechanics system. It is required that the quantum system obtained must go over into the original classical one in the limit $\hbar \to 0$ where $\hbar$ is Plank's constant. In recent times, a deformation quantification has bee explored in several context: in string theory approach to noncommutative geometry [1], Matrix Models [2], the noncommutative Yang-Mills theories [3] and non-commutative gauge theories [4].

Recently, the star product associated with an arbitrary two-dimensional Poisson structure, using the coherent states on the complex plane, was introduced [5]. It was shown that, from the coherent states adapted to harmonic oscillator, one recover easily the well-known Moyal star-product [6]. The deformed coherent states (à la Man'ko et al) [7] were also considered to provide an associative star-product. Then, it is clear now that the coherent state formulation give an useful scheme to define the star-product in a consistent way.

The approach taken in this work is along the lines of Berezin quantization [8] and relies on coherent states (Gazeau-Klauder(GK) [9] and Perelomov-Klauder,(PK) [10, 11]) adapted to an exact solvable systems with a nonlinear spectrum [12, 13]. The use of the coherent states is due to their useful property of overcompletness. For our purpose, we will consider the coherent states à la Gazeau-Klauder and ones defined following Perelomov-Klauder approach. Theses constructions leads as we will see to inequivalent states except for the harmonic oscillator case.

We start by introducing the creation and annihilation operators corresponding to a quantum systems with non-linear spectrums of type $e_n = an^2 + bn \quad (n \in \mathbf{N}, a \geq 0, b > 0)$. For some particular values of a and b, one find again well-known quantum mechanical systems like Pöschl-Teller potential(see [12]), x^4-anharmonic oscillator [14], and standard harmonic oscillator. Section 3 is devoted to the construction of Gazeau-Klauder and Perelomov-Klauder coherent states for the above non-linear quantum systems. Their properties (resolution to unity and analytical representations) are also presented. In section 4, the Gazeau-Klauder coherent states (eigenstates of the annihilation operator) leads easily to the definition of star-product and Moyal bracket on the complex plane. However, when one deals with PK coherent states, the previous definition becomes nontrivial due to the fact that this kind of coherent states are not eigenstates of the annihilation operator. To overcome this difficulty, we introduce a new annihilation operator that diagonalizes the PK states. Concluding remarks are given in the last section.

2 Non-linear quantum spectrums

Choose a Hamiltonian H with a discrete spectrum which is bounded below, and has been adjusted so that $H \geq 0$. We assume that the eigenvalues of H are non-degenerate. The eigenstates $|\psi_n>$ of H are orthonormal vectors and they satisfy

$$H|\psi_n >= e_n|\psi_n > \tag{1}$$

We suppose that $e_n \geq 0$ and verifies $e_{n+1} > e_n$. The energy e_0 of the ground state $|\psi_0>$ is chosen to be zero. It is well-known that for such system one can factorize the Hamiltonian H in terms of creation a^+ and annihilation a^- operators as follows

$$H = a^+a^- \tag{2}$$

Theses operators acts on the Hilbert space $\mathcal{H} = \{|\psi_n>, n \in \mathbf{N}\}$, as

$$a^+|\psi_n >= \sqrt{e_{n+1}}|\psi_{n+1}> \quad \text{and} \quad a^-|\psi_n >= \sqrt{e_n}|\psi_{n-1}>, \tag{3}$$

implemented by $a^-|\psi_0 >= 0$. We define the operator G

$$[a^-, a^+] = G \sim G(N) \tag{4}$$

as the commutator between a^- and a^+ . It is clear, from equation (3), that the action of G on the state $|\psi_n>$ is given by

$$G|\psi_n >= [a^-, a^+]|\psi_n >= (e_{n+1} - e_n)|\psi_n > \tag{5}$$

The operators N is defined such that

$$N|\psi_n >= n|\psi_n > \tag{6}$$

Note that in general the operator N is different from H. They coincides only in the harmonic oscillator case. Furthermore, one can verify also the following commutation relations

$$[N, a^\pm] = \pm a^\pm \tag{7}$$

In this letter, as we mentioned before, we focus our attention of a quantum systems with energy spectrum of type

$$e_n = an^2 + bn \quad , n = 0, 1, 2, ... \quad a \geq 0 \;\;,\;\; b > 0 \tag{8}$$

This choice covers many interesting situations. Indeed, for $(a = 1, b = k + k')$, we have the spectrum of a quantum system evolving in the Pöschl-Teller potentials parametrized by k and k' ($k > 1$ and $k' > 1$) [12, 13]

$$H = -\frac{d^2}{dx^2} + V(x) \tag{9}$$

where

$$V(x)=\frac{1}{4}(\frac{k(k-1)}{\sin^2(x/2)}+\frac{k'(k'-1)}{\cos^2(x/2)}) \quad 0<x<\pi \tag{10}$$

and $V(x)=\infty$ otherwise (i.e $x\geq 0 \quad ; \quad x\geq\pi$). This family of potentials is also called, sometime, the Pöschl-Teller potentials of the first kind. The latter reduces to other interesting potentials, which are widely used in solid state and molecular physical, like for instance Scarf and Rosen-Morse ones ([12] and references quoted therein). The case ($a=1, b=k+k'=2$), correspond to the spectrum of a free particle trapped in the infinite square-well potential. In the case ($a=\frac{3\epsilon}{2}, b=a+1$), where the parameter ϵ is positive, we have the energy levels of so-called the x^4-anharmonic oscillator [14] describing by the Hamiltonian

$$H=a_0^+a_0^-+\frac{\epsilon}{4}(a_0^-+a_0^+)^4-c_0 \tag{11}$$

where $c_0=\frac{3\epsilon}{4}-\frac{21\epsilon^2}{2}$ and a_0^+ , a_0^- are annihilation and creation operator ($\{a_0^-,a_0^+\}=1$) of the harmonic oscillator. This quantum system has been extensively studied since the early 1970 (see review [14]) . Finally, for $a=0$ and $b=1$, we obtain the standard harmonic oscillator spectrum which can be also obtained from x^4-anharmonic system in the limit $\epsilon\to 0$.

3 Coherent states

Coherent states play an important role in many different context of theoretical and experimental physics, especially quantum optics [11]. This notion was firstly discovered for the harmonic oscillator and has been extended for several other potentials in the references [9, 12, 13] in which coherent states are defined: (i) as eigenstates of the annihilation operator, (ii) by acting the displacement operator on the ground state $|\psi_0>$ and (iii) as states minimizing the so-called the Robertson-Shrodinger uncertainty relation. The definitions (i),(ii),(iii) gives different sets of states when one deal with a quantum system other than the harmonic oscillator. As we have mentioned above, we investigate the way to construct the star-product using the coherent states associated with an arbitrary quantum systems having spectrum of type $e_n=an^2+bn$. Two types of coherent states will be used. The first set is the so-called Gazeau-Klauder GK coherent states obtained from the definition (i). The second type are of Perelomov-Klauder PK constructed following the definition (ii). Note that the minimization of the Robertson-Scrodinger uncertainty relation leads to the so-called generalized intelligent states, which are not of interest in this work

3.1 Gazeau-Klauder Coherent states

Let us denote the Gazeau-Klauder coherent states by $|z>, z\in\mathbf{C}$. They are defined as eigenstates of the annihilation operator a^-

$$a^-|z>=z|z> \tag{12}$$

Decomposing $|z>$ in Hilbert space $\mathcal{H}$ basis, and using the action of a^- on the $|\psi_n>$'s given by (3), we show that the coherent states $|z>$ are as follows

$$|z>=\mathcal{N}(|z|^2)^{-1}\sum_{n=0}^{\infty}\frac{(\Gamma(r+1))^{1/2}z^n}{n!(\Gamma(n+r+1))^{1/2}a^{n/2}}|\psi_n> \tag{13}$$

where $r=\frac{b}{a}$ and the normalization constant $\mathcal{N}(|z|^2)$ is

$$(\mathcal{N}(|z|^2))^2={}_0F_1(r+1,\frac{|z|^2}{a}) \tag{14}$$

The set of states $|z>$ is overcomplete. Indeed, the resolution of unity

$$\int |z><\bar{z}|d\mu(z,\bar{z})=I_{\mathcal{H}} \tag{15}$$

is ensured in respect to the measure :

$$d\mu(z,\bar{z})=\frac{2}{\pi a}I_r(\frac{2r}{\sqrt{a}})K_{r/2}(\frac{2r}{\sqrt{a}})rdrd\theta \quad , \quad z=re^{i\theta} \tag{16}$$

The latter formula can be determined in a different ways. Here, we have used the approach developed in [12, 13]. The kernel (overlapping of two coherent states) is given by

$$<z'|z>=\frac{{}_0F_1(r+1,\frac{\bar{z}'z}{a})}{({}_0F_1(r+1,\frac{|z|^2}{a}){}_0F_1(r+1,\frac{|z'|^2}{a}))^2} \tag{17}$$

The overcompletion of the set $\{\ |z>, z\in\mathbf{C}\ \}$ provide a representation of any state by the entire function

$$f(z)=({}_0F_1(r+1,\frac{|z|^2}{a}))^{1/2}<\bar{z}|f> \tag{18}$$

In particular, the analytic function corresponding the eigenstates $|\psi_n>$ are

$$\mathcal{F}_n(z)=\frac{z^n\sqrt{\Gamma(r+1)}}{a^{n/2}(n!\Gamma(n+r+1))^{1/2}} \tag{19}$$

On the set $\{\mathcal{F}_n(z)\}$ the action of the creation and annihilation operators are given by

$$a^+=z \quad , \quad a^-=(z\frac{d^2}{dz^2}+(r+1))\frac{d}{dz}, \tag{20}$$

and the operator G acts as

$$G=2az\frac{d}{dz}+(a+b) \tag{21}$$

It is easy to see that theses operators act in the functions space $\{\mathcal{F}_n(z), n\in\mathbf{N}\}$ as

$$a^+\mathcal{F}_n(z)=\sqrt{e_{n+1}}\mathcal{F}_{n+1}(z) \quad a^-\mathcal{F}_n(z)=\sqrt{e_n}\mathcal{F}_{n-1}(z) \quad G\mathcal{F}_n(z)=(e_{n+1}-e_n)\mathcal{F}_n(z) \tag{22}$$

This realization will be useful in the sequel of this work when we will introduce the star-product approach based on the Gazeau-Klauder Coherent states.

3.2 Perelomov-Klauder Coherent states

We recall that the Perelomov-Klauder Coherent are defined by :

$$|z>= \mathcal{D}(z)|\psi_0>= exp(za^+ - \bar{z}a^-)|\psi_0> \tag{23}$$

The computation of the action of the displacement operator $\mathcal{D}(z)$ on the ground state $|\psi_0>$ was done for an arbitrary quantum system and illustrated for the Pöschl-Teller potentials [13]. Note that this result can be also applied for a quantum systems possessing energy levels $e_n = an^2 + bn$ $(n \in \mathbf{N})$ with a minor modifications .Then, one can obtain

$$|\zeta>= (1-|\zeta|^2)^{\frac{r+1}{2}} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+r+1)}{n!\Gamma(r+1)}} \zeta^n |\psi_n> \tag{24}$$

where $\zeta = \frac{z}{|z|}\tanh(z\sqrt{a})$. The states $|\zeta>$ satisfies the resolution to unity, namely

$$\int |\zeta><\zeta| d\mu(\zeta,\bar{\zeta}) = I_{\mathcal{H}}, \tag{25}$$

in respect to the measure given by

$$d\mu(\zeta,\bar{\zeta}) = \frac{r}{\pi}\frac{d^2\zeta}{(1-|\zeta|^2)^2} \tag{26}$$

The kernel $<\zeta'|\zeta>$ is given by

$$<\zeta'|\zeta>= (1-|\zeta'|^2)^{\frac{r+1}{2}}(1-|\zeta|^2)^{\frac{r+1}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+r+1)}{n!\Gamma(r+1)}(\bar{\zeta}'\zeta)^n \tag{27}$$

The state $|\psi_n>$ is represented analytically by the function

$$\mathcal{G}_n(\zeta) = \zeta^n \sqrt{\frac{\Gamma(n+r+1)}{n!\Gamma(r+1)}} \tag{28}$$

The creation and annihilation operators act in the Hilbert space of analytical functions $\{\mathcal{G}_n(\zeta), n \in \mathbf{N}\}$ as a first order differential operators

$$a^+ = \zeta^2\frac{d}{d\zeta} + (r+1)\zeta \quad , \quad a^- = \frac{d}{d\zeta} \tag{29}$$

and the operator G acts in the same representation as

$$G = 2\zeta\frac{d}{d\zeta} + (r+1) \tag{30}$$

One can verify that

$$a^+\mathcal{G}_n(\zeta) = \sqrt{e_{n+1}}\mathcal{G}_{n+1}(\zeta) \quad a^-\mathcal{G}_n(\zeta) = \sqrt{e_n}\mathcal{G}_{n-1}(\zeta) \quad G\mathcal{G}_n(\zeta) = e_n\mathcal{G}_n(\zeta) \tag{31}$$

To end this subsection, we would like to draw the attention that the analytical representations of both the (GK) and (PK) coherent states are related through the Laplace transform [15]

4 Star product and Moyal Bracket

In this section, we introduce the star-product and Moyal bracket in coherent states framework. Let us start by recalling the definition of star-product. To every operator A acting on the Hilbert space $\mathcal{H}$ one can associate a function $\mathcal{A}(z,\bar{z})$ on the complex plane as

$$\mathcal{A}(z,\bar{z}) =< z|A|z > \tag{32}$$

The associative star-product of two functions $\mathcal{A}(z,\bar{z})$ and $\mathcal{B}(z,\bar{z})$ is defined by [5]

$$\mathcal{A}(z,\bar{z}) * \mathcal{B}(z,\bar{z}) =< z|AB|z > \tag{33}$$

and then the corresponding Moyal bracket is given by

$$\{\mathcal{A}(z,\bar{z}),\mathcal{B}(z,\bar{z})\}_M = \mathcal{A}(z,\bar{z}) \star \mathcal{B}(z,\bar{z}) - \mathcal{B}(z,\bar{z}) \star \mathcal{A}(z,\bar{z}) =< z|[A,B]|z > . \tag{34}$$

Using the identity resolution of the coherent states, the star-product equation (33) becomes

$$\mathcal{A}(z,\bar{z}) \star \mathcal{B}(z,\bar{z}) = \int d\mu(\zeta,\bar{\zeta}) < z|A|\zeta >< \zeta|B|z >, \tag{35}$$

which can be also written as

$$\mathcal{A}(z,\bar{z}) \star \mathcal{B}(z,\bar{z}) = \sum_{n,m} < z|\psi_n >< \psi_n|AB|\psi_m >< \psi_m|z > \tag{36}$$

in terms of the function $< \psi_m|z >$ corresponding to the element $|\psi_m >$ of the Hilbert space $\mathcal{H}$. It follows that the Moyal bracket take the form

$$\{\mathcal{A}(z,\bar{z}),\mathcal{B}(z,\bar{z})\}_M = \sum_{n,m} < z|\psi_n >< \psi_n|[A,B]|\psi_m >< \psi_m|z > \tag{37}$$

Analysing the relations (37), we see that there is a correspondence between the structure relations of the operators algebra of the and the $\star$ commutators, namely Moyal bracket, of the elements generating the algebra of the functions on the complex plane. This point will be examined through this this section.
We note that the star-product (33) can be written in the integral representation in terms of the ordered exponential [5] :

$$\star = \int d\mu(\zeta,\bar{\zeta}) : \exp(\frac{\overrightarrow{\partial}}{\partial\eta}(\zeta-\eta) : | < \eta|\zeta > |^2 : \exp((\bar{\zeta}-\bar{\eta})\frac{\overleftarrow{\partial}}{\partial\bar{\eta}} : \tag{38}$$

which not used in this work.

4.1 Star product with Gazeau-Klauder Coherent states

In the GK Coherent states, the star-product take the simple form

$$\mathcal{A}(z,\bar{z}) \star \mathcal{B}(z,\bar{z}) = \mathcal{N}(|z|^2)^{-2} \sum_{n,m} \mathcal{F}_n(\bar{z}) < \psi_n|AB|\psi_m > \mathcal{F}_m(z) \tag{39}$$

Since the GK coherent states are the eigenstates of the annihilation operators a^-, there are a correspondence between a^- and the analytic function $z \to z$

$$< z|a^-|z >= z \tag{40}$$

Then, the anti-analytic function $z \to \bar{z}$ is the expectation value of the operators a^+ over the coherent state $|z >$

$$< z|a^+|z >= \bar{z} \tag{41}$$

Furthermore, by using the definition of the star-product (39) one can obtain easily the following relations

$$1 \star 1 = 1 \qquad 1 \star z = z \star 1 = z \qquad 1 \star \bar{z} = \bar{z} \star 1 = \bar{z} \tag{42}$$

and more generally, we have

$$z^{\star p} = z^p \qquad \bar{z}^{\star p} = \bar{z}^p \qquad p \geq 0 \tag{43}$$

where $\theta^{\star p} = \theta \star \theta \star \theta ... \star \theta$ (p times, with $\theta = z$ or $\bar{z}$). Based on the latter relations, one can evaluated the star-product for any z-analytics and $\bar{z}$-antianalytics functions which are given by

$$\mathcal{A}(z) \star \mathcal{B}(z) = \mathcal{A}(z)\mathcal{B}(z) \tag{44}$$

and

$$\mathcal{A}(\bar{z}) \star \mathcal{B}(\bar{z}) = \mathcal{A}(\bar{z})\mathcal{B}(\bar{z}) \tag{45}$$

We show also that

$$\bar{z} \star z =< z|a^+a^-|z >= z\bar{z} = |z|^2 \tag{46}$$

Hence the star-product between two functions is reduced to the ordinary one if the function in the right is analytic and the function in the left is anti-analytic

$$\mathcal{A}(\bar{z}) \star \mathcal{B}(z) = \mathcal{A}(\bar{z})\mathcal{B}(z) \tag{47}$$

For completeness, we will compute the star product of type $z \star \bar{z}$. From the previous considerations it is easy to see that

$$z \star \bar{z} =< z|a^-a^+|z >= \bar{z} \star z - \mathcal{G}(z,\bar{z}). \tag{48}$$

where

$$\mathcal{G}(z,\bar{z}) = \frac{2a|z|^2}{r+1}\frac{{}_0F_1(r+2,\frac{|z|^2}{a})}{{}_0F_1(r+1,\frac{|z|^2}{a})} + (a+b) \tag{49}$$

The function $\mathcal{G}(z,\bar{z})$ can be expressed as follows

$$\mathcal{G}(z,\bar{z}) = 2az\frac{d}{dz}\sum_n \mathcal{F}_n(\bar{z})\mathcal{F}_n(z) + a + b \tag{50}$$

in terms of the functions $\{\mathcal{F}_n(z), n \in \mathbf{N}\}$.
One remark that the Moyal bracket preserve the commutation relations of the algebra generated by $\{a^-, a^+, G\}$ as we have already mentioned . Indeed, in the operators language we have the following relations

$$[a^-, a^+] = G(N) \quad , \quad [a^\pm, G(N)] = \pm 2aa^\pm \tag{51}$$

which are are expressed in the language of Moyal bracket as

$$\{z,\bar{z}\}_M = \mathcal{G}(z,\bar{z}) \quad , \quad \{z, \mathcal{G}(z,\bar{z})\}_M = 2az \quad , \quad \{\bar{z}, \mathcal{G}(z,\bar{z})\}_M = -2a\bar{z} \tag{52}$$

Using the previous results, one can show the following interesting relations

$$\begin{aligned}
\bar{z} \star \mathcal{G}(z,\bar{z}) &= \mathcal{G}(z,\bar{z}) + \bar{z}(a+b) - (a+b) \\
\mathcal{G}(z,\bar{z}) \star \bar{z} &= \mathcal{G}(z,\bar{z}) + \bar{z}(3a+b) - (a+b) \\
\mathcal{G}(z,\bar{z}) \star z &= \mathcal{G}(z,\bar{z}) + z(a+b) - (a+b) \\
z \star \mathcal{G}(z,\bar{z}) &= \mathcal{G}(z,\bar{z}) + z(3a+b) - (a+b)
\end{aligned} \tag{53}$$

which are useful to calculate in a complete way the star-product. Finally, using the relations (42), (43), (46), (52) and (53), one can compute the star-product of any two functions $\mathcal{A}(z,\bar{z})$ and $\mathcal{B}(z,\bar{z})$. As illustration, let us give the following example : the star-product of $\mathcal{A}(z,\bar{z}) = \bar{z}$ and $\mathcal{B}(z,\bar{z}) = \bar{z}z$ are given by

$$\begin{aligned}
\mathcal{A}(z,\bar{z}) \star \mathcal{B}(z,\bar{z}) &= \bar{z}^2 z \\
\mathcal{B}(z,\bar{z}) \star \mathcal{A}(z,\bar{z}) &= \bar{z}^2 z + \mathcal{G}(z,\bar{z}) + \bar{z}(a+b) - (a+b),
\end{aligned} \tag{54}$$

and the corresponding Moyal bracket is

$$\{\mathcal{A}(z,\bar{z}), \mathcal{B}(z,\bar{z})\}_M = (a+b) - \mathcal{G}(z,\bar{z}) - \bar{z}(a+b). \tag{55}$$

In the particular case : $a = 0$ and $b = 1$ (i.e., the harmonic oscillator case), the function $\mathcal{G}(z,\bar{z})$ is equal to unity and the relations (53) reduces to (42) ones. The relation (52) gives the well-known Moyal bracket constructed using the coherent adapted to standard

harmonic oscillator [5].

It is true that for the quantum systems considered in this work, we can define the Gazeau-Klauder coherent states as well-as the Perelomov-Klauder ones. However, it should be noted that there exist some quantum systems for which the Gazeau-Klauder coherent states can not be constructed due to the fact that the dimension of the Hilbert space $\mathcal{H}$ is finite like for instance a quantum system trapped in the Morse potential [16]. In this situation, the definition of the star-product discussed above can not be used. So, for this reason, we believe that is interesting to introduce also the star-product in the Perelomov-Klauder coherent states.

4.2 Star product with Perelomov-Klauder Coherent states

The Perelomov-Klauder Coherent states $|\zeta>$ equation (24) are not the eigenstates of the annihilation operator a^-. In order to define the star-product and Moyal bracket, one may ask if the analytic function $\zeta \to \zeta$ and the anti-analytic function $\zeta \to \bar{\zeta}$, can be defined as the means values of some operators A^- and A^+ acting on the Hilbert space $\mathcal{H}$:

$$< \zeta|A^-|\zeta >= \zeta \qquad < \zeta|A^+|\zeta >= \bar{\zeta} \tag{56}$$

Let us introduce the operators

$$A^- = a^- f(N) \qquad A^+ = f(N)a^+ \tag{57}$$

The operators A^- and A^+ satisfy the relations (56), when $f(N)$ is defined by

$$f(N) = \frac{N+1}{g(N+1)} \tag{58}$$

where the function operator $g(N+l)$ acts in the Hilbert space as $|\psi_n>$

$$g(N+l)|\psi_n >= e_{n+l}|\psi_n > \tag{59}$$

for $l \in \mathbf{N}$. The new operators A^- and A^+ satisfy the following relations

$$[A^-, A^+] = D(N) \tag{60}$$

where the operator $D(N)$ is defined as a function of the operator N by

$$D(N) = \frac{(N+1)^2}{g(N+1)} - \frac{N^2}{g(N)} \tag{61}$$

One can show also that

$$A^- D(N) = D(N+1)A^- \qquad A^+ D(N) = D(N-1)A^+ \tag{62}$$

For our purpose, we define the following functions

$$\mathcal{D}_l(\zeta,\bar{\zeta}) =< \zeta|D(N+l)|\zeta > \tag{63}$$

which are useful, in the computation of the star-product based on the Perelomov-Klauder coherent states. A straightfoward calculation leads to

$$\mathcal{D}_l(\zeta,\bar{\zeta}) = (1-|\zeta|^2)^{r+1}\sum_n \mathcal{G}_n(\bar{\zeta})\mathcal{G}_n(\zeta)\{(n+l+2)^2\frac{e_{n+l+1}}{e^2_{n+l+2}} - (n+l+1)^2\frac{e_{n+l}}{e^2_{n+l+1}}\} \tag{64}$$

Using (64), one find the basic relations needed for a computation of star-product between any two functions $\mathcal{A}_l(\zeta,\bar{\zeta})$ and $\mathcal{B}_l(\zeta,\bar{\zeta})$. They are given

$$\begin{array}{ll} 1\star\zeta = \zeta\star 1 = \zeta & 1\star\bar{\zeta} = \bar{\zeta}\star 1 = \bar{\zeta} \\ \bar{\zeta}\star\zeta = \bar{\zeta}\zeta & \zeta\star\bar{\zeta} = \bar{\zeta}\zeta + \mathcal{D}_0(\zeta,\bar{\zeta}) \\ \zeta\star\mathcal{D}_l(\zeta,\bar{\zeta}) = \zeta\mathcal{D}_{l+1}(\zeta,\bar{\zeta}) & \bar{\zeta}\star\mathcal{D}_l(\zeta,\bar{\zeta}) = \bar{\zeta}\mathcal{D}_l(\zeta,\bar{\zeta}) \\ \mathcal{D}_l(\zeta,\bar{\zeta})\star\zeta = \zeta\mathcal{D}_l(\zeta,\bar{\zeta}) & \mathcal{D}_l(\zeta,\bar{\zeta})\star\bar{\zeta} = \bar{\zeta}\mathcal{D}_{l+1}(\zeta,\bar{\zeta}) \end{array} \tag{65}$$

As application, we set $\mathcal{A}(\zeta,\bar{\zeta}) = \bar{\zeta}$ and $\mathcal{B}(\zeta,\bar{\zeta}) = \bar{\zeta}\zeta$. The star-product is this case are given by

$$\begin{array}{c} \mathcal{A}(\zeta,\bar{\zeta})\star\mathcal{B}(\zeta,\bar{\zeta}) = \bar{\zeta}^2\zeta \\ \mathcal{B}(\zeta,\bar{\zeta})\star\mathcal{A}(\zeta,\bar{\zeta}) = \bar{\zeta}^2\zeta + \bar{\zeta}\mathcal{D}_0(\zeta,\bar{\zeta}) \end{array} \tag{66}$$

where $\mathcal{D}_0(\zeta,\bar{\zeta})$ is defined by (64).
Contrary to the previous case (one corresponding to GK coherent states), the structure relations of the algebra $\{a^+, a^-, G\}$ are not preserved by this star-product. However, one can see that the Moyal bracket defined from Perelomov-Klauder coherent states preserve the commutation relations (60) and (62) of the algebra generated by $\{A^+, A^-, D(N)\}$.

For the harmonic oscillator case ($a = 0, b = 1$), we have $f(N) = 1$, $D(N) = 1$ and the operators $A^\pm$ reduce to the creation and annihilation operators of ordinary harmonic oscillator where the Moyal bracket is trivial.

We conclude that the star product in GK coherent states is in general different from one obtained in the PK scheme, except for the ordinary oscillator case.

5 Concluding Remarks

In this work, we have introduced the star product and the Moyal bracket in the coherent states framework corresponding to exact solvable quantum systems, admitting a

nonlinear spectra. We have seen that in the PK coherent states case, the construction becomes non-trivial, because the coherent states are not eigenstates of the annihilation operator. This difficulty was removing by introducing a new operator diagonalizing the PK states. The fundamentals star-products, providing a complete way to compute the Moyal bracket for any two functions, are given in this work. The star product constructed for the standard harmonic oscillator [5] was recorded as the particular case of our approach. It is clear that there remain many problems for future study. One of them would be the definition of star-product using the coherent states for the Lie algebras and their supersymmetric counterparts. Another would be a better understanding of the relationship between the star product with GK coherent states and one using the PK ones. We believe that such relation can be established, because as we mentioned above, the analytical representations of both coherent states are related through Laplace transformation. Finally, it became apparent from this work that the construction of the star product from the coherent states involves a certain rule of correspondence between functions on non-commuting operators and analytical functions; this correspondence is similar to one between classical and quantum mechanics. So, would be interesting, as suggested by one of the referee of this paper, to show this calculus in use by studying an exactly solvable quantum mechanics system cited in this work (a system trapped in Pöschl-Teller potential, for instance). This matter in under consideration [17]

Acknowledgments

The authors would like to thank the Abdus Salam International Center for theoretical Physics, Trieste, Italy, for hospitality. They also would like to thank the referees for their suggestions and remakes.

References

[1] C.S. Chu and P.M. Ho, Nucl.Phys.B **550** 151 (1999).

[2] A. Connes, M.R. Douglas and A. Schwarz, Deformation Quantization and Matrix theory: Compactification on tori, JHEP 9802/003, hep-th/9711162, M.R.Douglas and C.Hull, D-branes and the noncommutative torus, JHEP 9802/008, hep-th/9711165

[3] B. Jurco and P. Schupp, Eur.Phy.J.C **14,** 367 (2000).

[4] T. Asakawa and I .Kishimoto, hep-th/0002138.

[5] G. Alexanian, A. Pinzul and A.Stern, Nucl.Phys.B **600** 531(2001).

[6] J. Moyal, Proc. Camb.Phil.Soc. **45,** 99(1949).

[7] V.I. Man'ko, G. Marmo, E.C.G. Sudarshan and F.Zaccaria, Physica.Scripta.**55,** 528 (1997).

[8] F.A. Berezin, Commun.Math.Phys. **40,** 153 (1975).

[9] J.P. Gazeau and J.P. Klauder, J.Phys.A: Math.Gen. **32,** 123 (1999).

[10] A. Perelomov, Generalized Coherent States and Their applications, Springer, Berlin (1985)

[11] J.R. Klauder and B.S. Skagerstam Coherent States, World Scientific, Singapor (1985)

[12] J.P. Antoine, J.P. Gazeau, P.M. Monceau, J.R. Klauder and K.A Penson, J.Math.Phys. **42** 2349 (2001).

[13] A.H. El Kinani and M. Daoud, J. Phys.A: Math.Gen. **34,** 5373 (2001), Phys.Lett.A **283,** 291 (2001), Int.J.Mod.Phys.B **15,** 2465 (2001).

[14] A.D. Speliotopoulos, J. Phys.A: Math.Gen. **33,** 3809 (2000).

[15] C. Brif, A.Vourdas and A Mann, J. Phys.A: Math.Gen. **29,** 5873 (1996).

[16] L.D. Landau and E. M. Lifshitz, Quantum mechanics, Pergamon (1965).

[17] M. Daoud and E.H. El Kinani 2002, in preparation.